Stat 201: Introduction to Statistics

Standard 19: Probability Distributions – Binomial

Let's Apply This to Categorical Variables: **The Binomial Distribution**

- We look at a categorical variable with two outcomes
  - We consider one a success and zero a failure

x		P(x)
Success (denoted as 1)	This is what we're interested in, even if it isn't particularly successful in the sense of the English word	p = Probability of a 'success'
Failure (denoted as 0)	This is the other case – what we aren't interested in ,even if it isn't particularly a failure in the sense of the English word	q = Probability of a 'failure' = 1- p

The Binomial Distribution

- **The Binomial Distribution Assumptions**
  1. It consists of **n trials** with **binary output**
     - They are denoted 1 or 0, or success and failure
  2. The probability of success on each trial is the same
     - The trials are **identical**
  3. The outcome of one trial does not affect the outcome of another trial
     - The trials are **independent**
  4. The binomial random variable x is the number of times we see a success in n trials

The Binomial Distribution: Notation

- **n =** the number of trials
- **p =** the probability of success for any given trial (this will be the same for every trial)
- **q =** the probability of failure for any given trial
  - By complement rule: $q = 1 - p$
- **X =** the number of successes for n trials
- **X** is the random variable, **n** and **p** are parameters; **x** will be the observation

Binomial Formula

- $P(X = x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$
- Recall: n! = n*(n-1)*(n-2)*…*2*1
  - Examples
    - 5! = 5*4*3*2*1=120
    - 0!=1
    - 5!/3!= 5*4

Binomial Calculations

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x} = binompdf(n, p, x)$

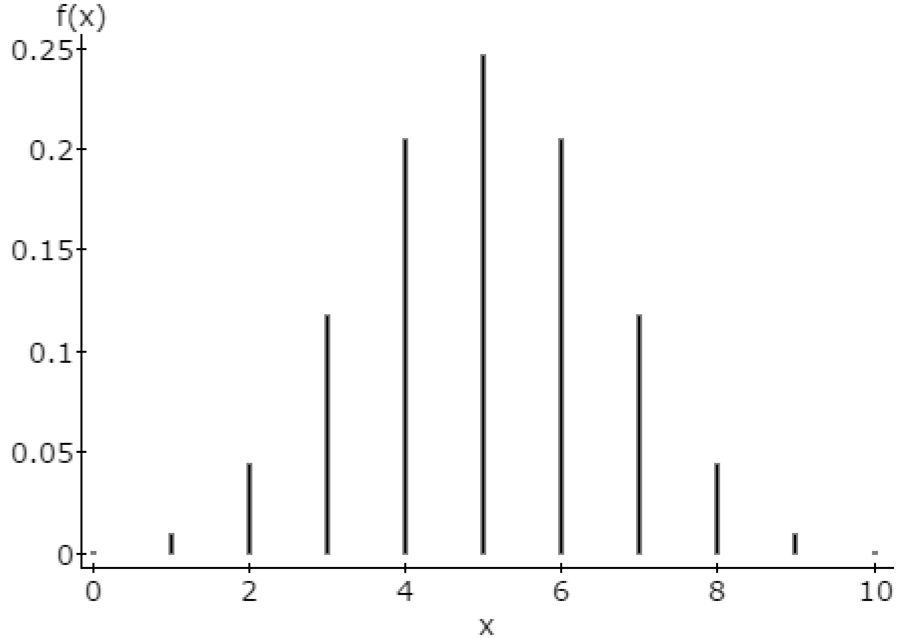
- $P(X \leq x) = \mathrm{P(X = x) + P(X = x - 1) + \cdots + P(X = 0) = binomcdf(n, p, x)}$

- $P(X > x) = 1 - \mathrm{P(X \leq x)} = 1 - \mathrm{binomcdf(n, p, x)}$

- PDF gives us probability of one point, the CDF gives us the probability of that point and everything less.
  - Recall cumulative frequency from Chapter 2

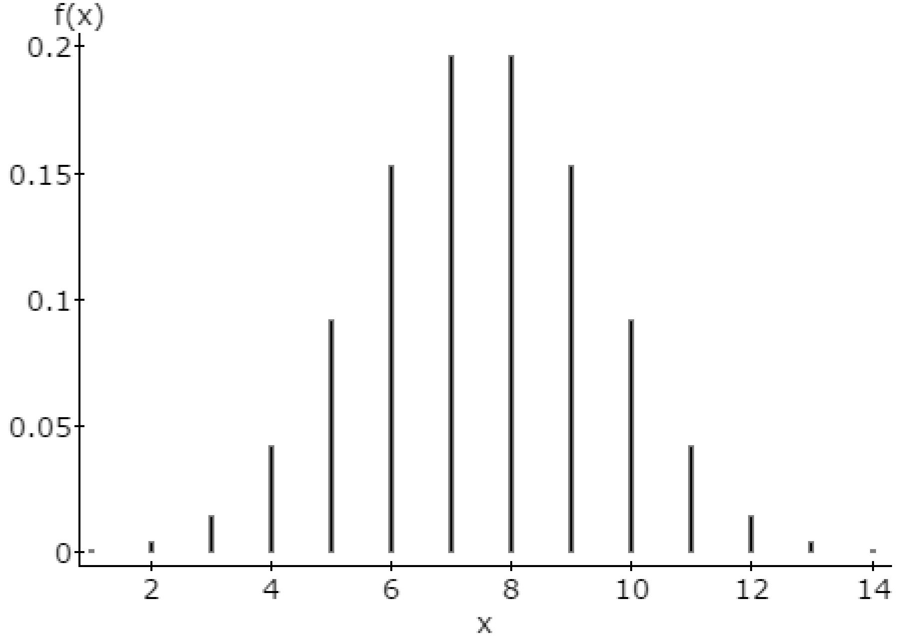
Shape of Binomial

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$
  - n=10, p=.5 : Bell shaped, but there’s empty space



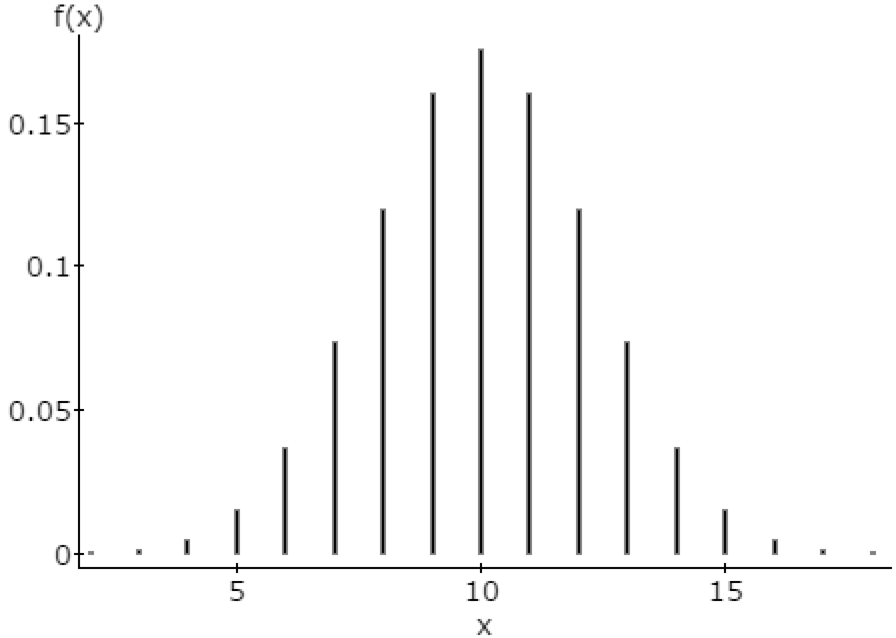
Shape of Binomial

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$
  - n=15, p=.5 : Bell shaped, but there's still empty space



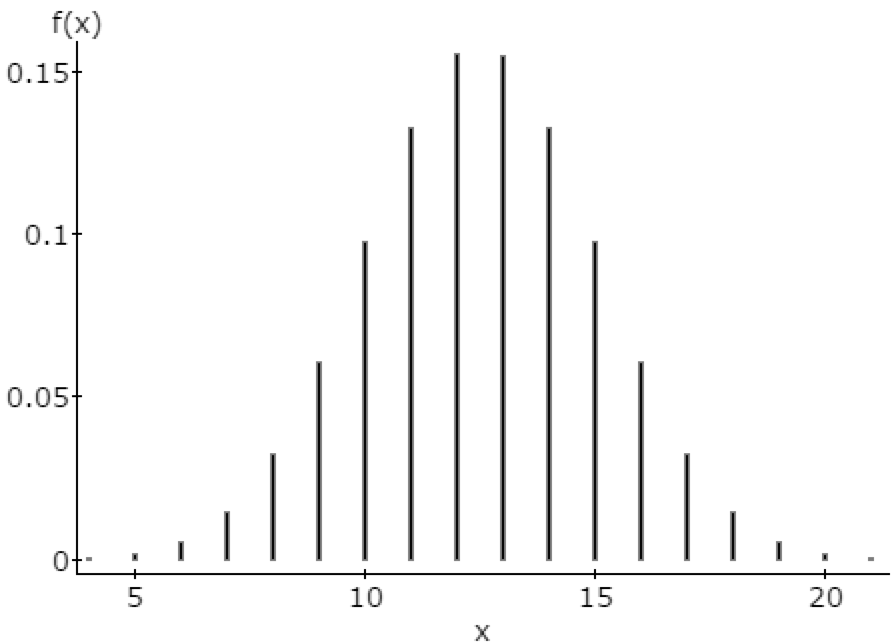
Shape of Binomial

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$
  - n=20, p=.5 : Bell shaped, but there's still empty space



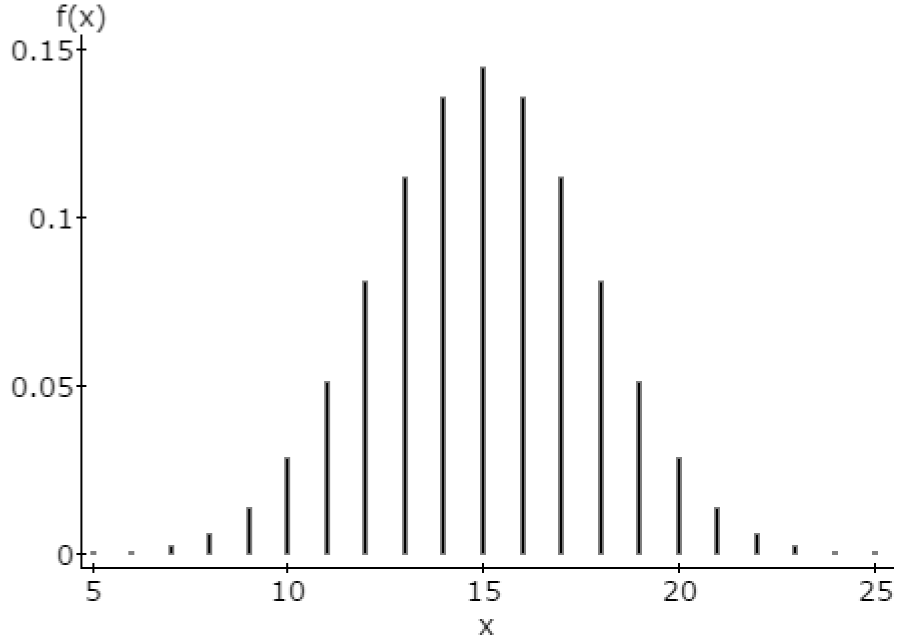
Shape of Binomial

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$
  - n=25, p=.5 : Bell shaped, but there's still empty space



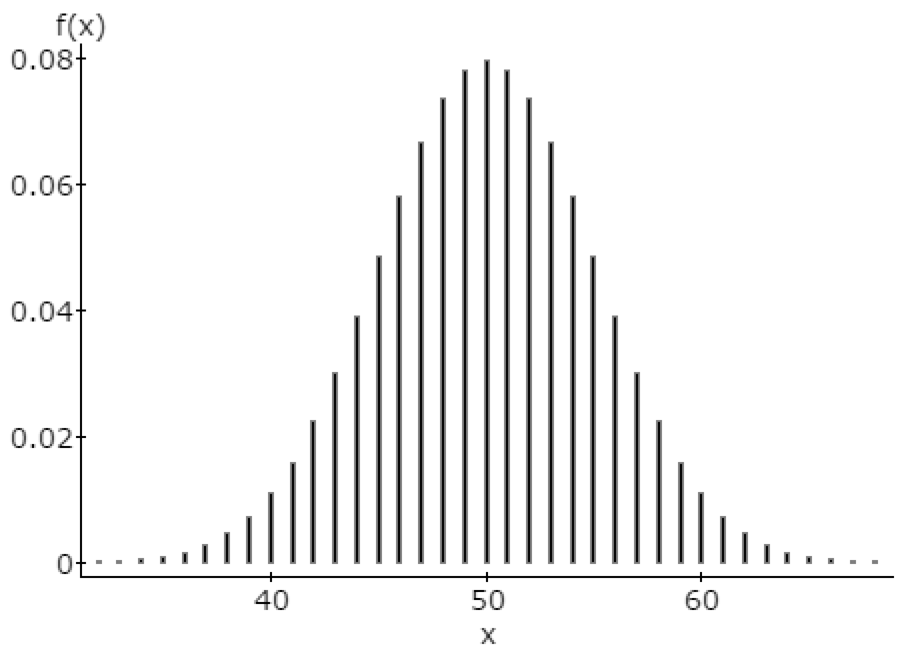
Shape of Binomial

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$
  - n=30, p=.5 : Bell shaped, but there's still empty space



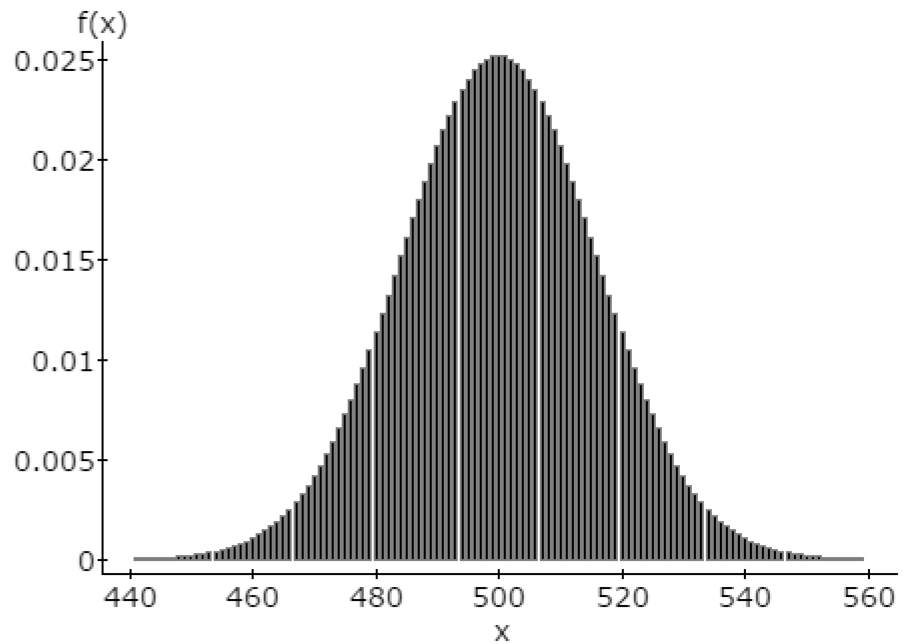
Shape of Binomial

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$
  - n=100, p=.5 : Bell shaped, but there's still empty space



Shape of Binomial

- $P(X=x)=\frac{n!}{x!(n-x)!}p^{x}q^{n-x}$
  - n=1000, p=.5 : Bell shaped, and space is negligible



Shape of Binomial

- $P(X = x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$

  - We will say that the binomial is bell-shaped if $n * p \geq 15 \; AND \; n * (1 - p) \geq 15$

  - We will say that the binomial is not bell-shaped if either $n * p < 15 \; OR \; n * (1 - p) < 15$

Shape of Binomial for Graphs

n	p	n*p	n*(1-p)	Bell-Shaped?
10	.5	10*.5 = 5 $<$ 15	10*(1-.5) = 5 $<$ 15	No
15	.5	15*.5 = 7.5 $<$ 15	15*(1-.5) = 7.5 $<$ 15	No
20	.5	20*.5 = 10 $<$ 15	20*(1-.5) = 10 $<$ 15	No
25	.5	25*.5 = 12.5 $<$ 15	25*(1-.5) = 12.5 $<$ 15	No
30	.5	30*.5 = 15 $\geq$ 15	30*(1-.5) = 15$\geq$ 15	**Yes**
100	.5	100*.5 = 50 $\geq$ 15	100*(1-.5) = 50$\geq$ 15	**Yes**
1000	.5	1000*.5 = 500$\geq$ 15	1000*(1-.5) = 500$\geq$ 15	**Yes**

Shape of More Complicated Binomials

n	p	n*p	n*(1-p)	Bell-Shaped?
10	.25	10*.25 = 2.5 < 15	10*(1-.25) = 7.5 < 15	No
15	.25	15*.25 = 3.75 < 15	15*(1-.25) = 11.25 < 15	No
20	.25	20*.25 = 5 < 15	20*(1-.25) = 15≥ 15	No
25	.25	25*.25 = 6.25 < 15	25*(1-.25) = 18.75 ≥ 15	No
30	.25	30*.25 = 7.5 < 15	30*(1-.25) = 22.5 ≥ 15	No
100	.25	100*.25 = 25 ≥ 15	100*(1-.25) = 75 ≥ 15	**Yes**
1000	.25	1000*.25 = 250≥ 15	1000*(1-.25) = 750 ≥ 15	**Yes**

Shape of a binomial

- For fixed p, as the sample size increases the probability distribution of X becomes bell shaped.
  - We consider n to be large enough when
  - $n * p > 15 \ AND \ \mathrm{n} * (1 - p) \geq 15$
  - This will be very important as we transition to inferential statistics.

What Sample Size Do I Need?

- Say we have that the probability of a success is .45, i.e. p=.45. What sample size would we need to have to say that the binomial is bell-shaped?

$$np \geq 15$$
$$n(.45) \geq 15$$
$$n \geq \frac{15}{.45}$$
$$n \geq 33.3333$$

<u>AND</u>

$$n(1-p) \geq 15$$
$$n(1-.45) \geq 15$$
$$n(.55) \geq 15$$
$$n \geq \frac{15}{.55}$$
$$n \geq 27.2727$$

- So, in order for both to be bigger than or equal to 15 we would need $n \geq 34$

Binomial Experiment – Example 1

- The two New England natives who founded Portland Oregon, Asa Lovejoy of Boston and Francis Pettygrove of Portland, Maine, both wanted to name the new city after their respective hometowns
- They decided to make the decision based on a best two-out-of-three coin toss.
- Let's say Pettygrove chose heads

Binomial Experiment – Example 1

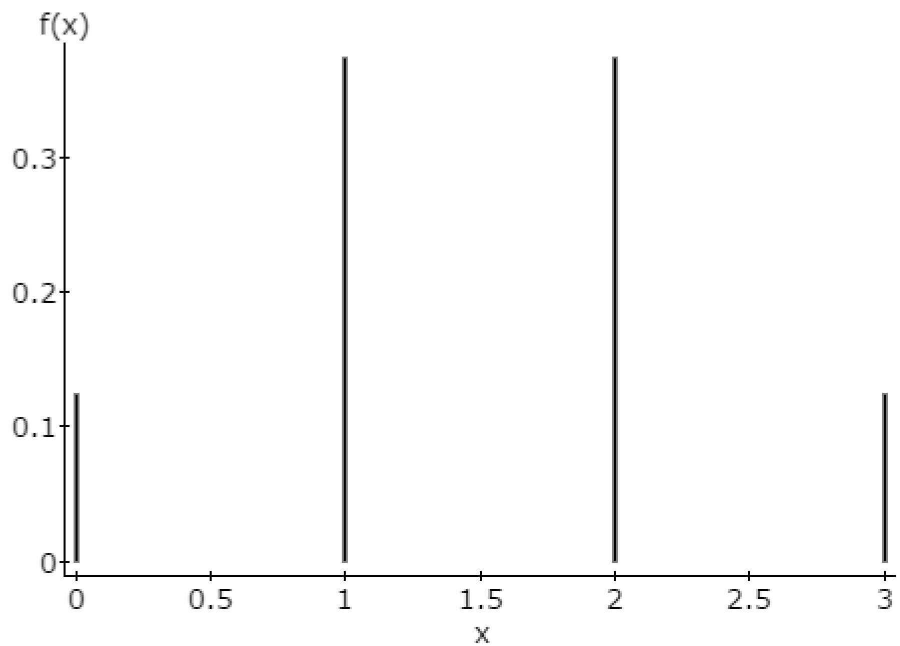
- A fair one-cent piece is flipped three times
  - Let X be the number of heads that occur
  - **n = 3**
  - p = .50
  - q = 1 - p = 1 - .50 =.50
- **Trials are identical** – we flip the same coin each time
- **Trials are independent** as the outcome of one trial doesn't affect another

Binomial Experiment – Example 1

- A fair one-cent piece is flipped three times
  - Let X be the number of heads that occur
  - **n = 3**
  - p = .50
  - q = 1 - p = 1 - .50 =.50
  - $np = 3 * .5 = 1.5 < 15$ and $n(1 - p) = 3 * (1 - .5) = 1.5 < 15$
  - Because $np < 15\ and\ n(1 - p) < 15$ we cannot say that the binomial is bell-shaped

Binomial Experiment – Example 1

- A fair one-cent piece is flipped three times
  - Because $np < 15\ and\ n(1-p) < 15$ we cannot say that the binomial is bell-shaped



Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times
  - Let X be the number of heads that occur
  - n = 3, p = .50, q =.50
  - Find the probability that there are exactly 2 heads

$$\mathrm{P}(\mathrm{X}=2)=\frac{n!}{x!\,(n-x)!}p^{x}q^{n-x}$$

$$=\frac{3!}{2!\,(3-2)!}(.5)^{2}(.5)^{3-2}=\frac{3!}{2!*1!}(.5)^{2}(.5)^{1}$$

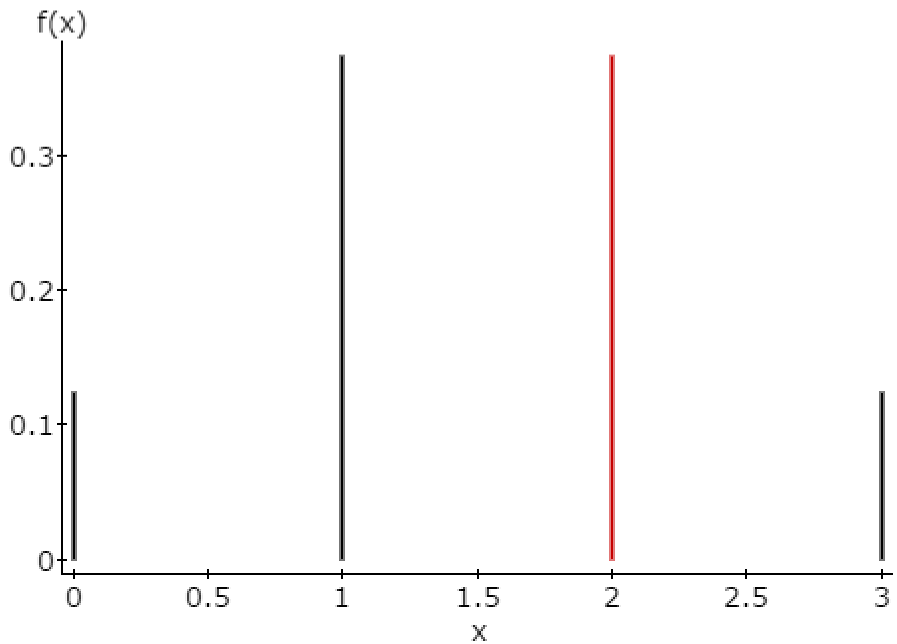
$$=\frac{3*2*1}{(2*1)*(1)}(.25)(.5)$$

$$=.375=binompdf(2,.5,2)$$

Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times

$$P(X = 2) = .375$$



Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times
  - Let X be the number of heads that occur
  - n = 3, p = .50, q =.50
  - Find the probability that there are exactly 3 heads

$$\mathrm{P(X = 3)} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \frac{3!}{3!\,(3-3)!} (.5)^3 (.5)^{3-3} = \frac{3!}{3! * 0!} (.5)^3 (.5)^0$$

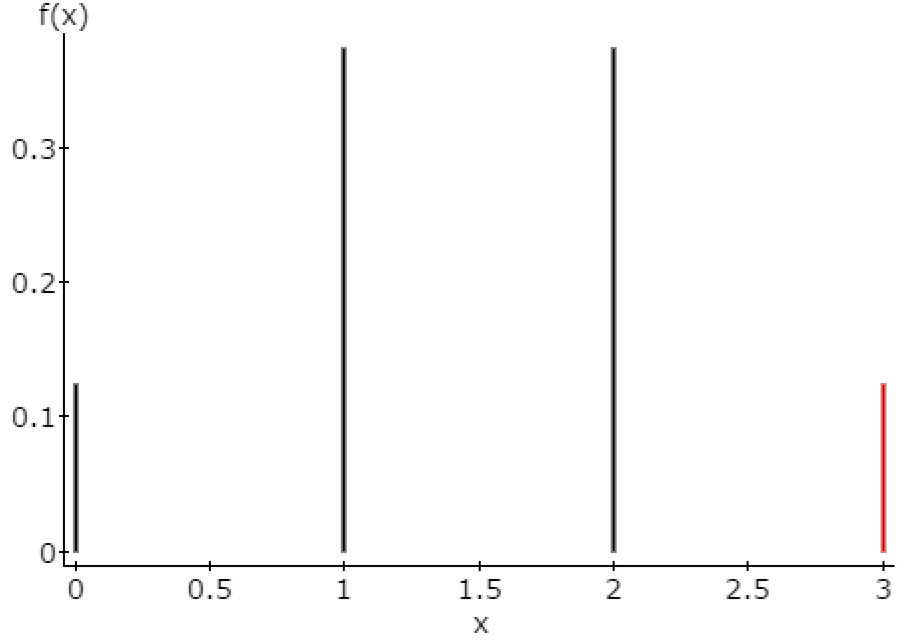
$$= \frac{3*2*1}{3*2*1} (.125)(1)$$

$$= .125 = binompdf(3, .5, 3)$$

Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times

$$P(X = 3) = .125$$



Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times
  - Let X be the number of heads that occur
  - n = 3, p = .50, q =.50
  - Find the probability that Pettygrove wins
    - i.e. Find the probability that there are at least 2 heads

$$\mathrm{P}(\mathrm{X} \geq 2) = P(X = 2) + P(X = 3) = .375 + .125 = .5$$

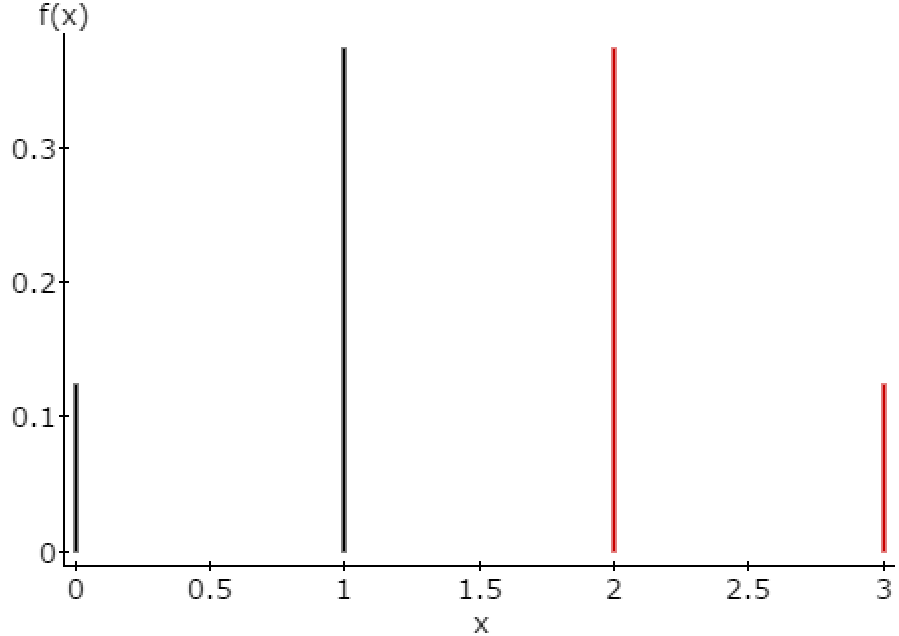
<u>OR Using Complement Rule</u>

$$\begin{aligned} \mathrm{P}(\mathrm{X} \geq 2) &= 1 - P(X < 2) \\ &= 1 - \left(P(X = 1) + P(X = 0)\right) \\ &= 1 - P(X \leq 1) \\ &= 1 - binomcdf(3, .5, 1) \end{aligned}$$

Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times

$$P(X \geq 2) = .5$$



Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times
  - Let X be the number of heads that occur
  - n = 3, p = .50, q =.50
  - Find the probability that Pettygrove loses(there are less than 2 heads)

$$\mathrm{P}(\mathrm{X} < 2) = 1 - \mathrm{P}(\mathrm{X} \geq 2)$$
$$= 1 - \left(P(X = 2) + P(X = 3)\right) = 1 - .5 = .5$$

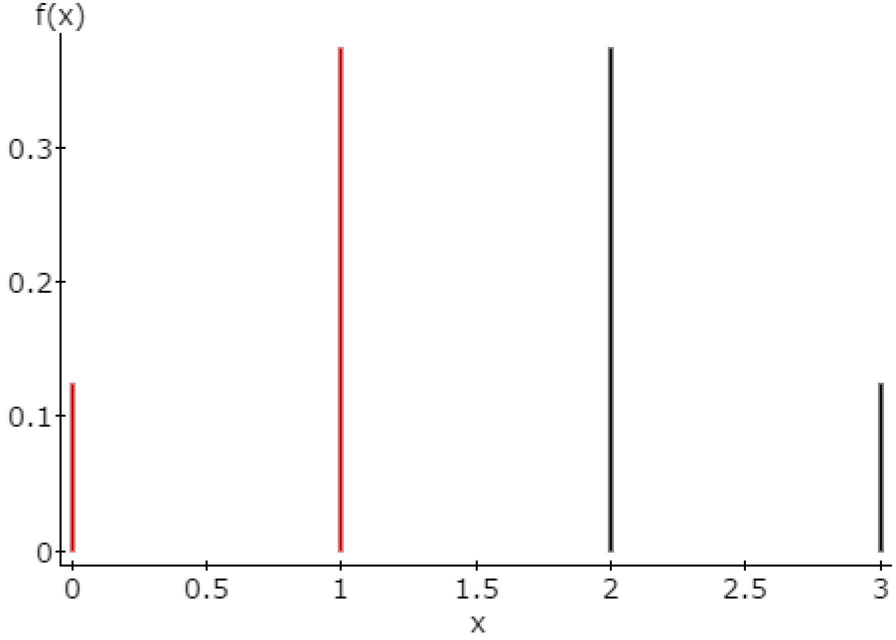
<u>OR Using Complement Rule</u>

$$P(X < 2) = P(X = 1) + P(X = 0)$$
$$= binomcdf(1, .5, 1) = .5$$

Binomial Experiment - Example 1

- A fair one-cent piece is flipped three times

$$P(X < 2) = .5$$



Binomial Experiment - Example 1

- The probability that Pettygrove wins $P(X \geq 2) = .5$
- The probability that Lovejoy wins

  $P(X < 2) = .5$
- We see that this is a **fair** game – they each have a 50% chance of winning
- So, why not just flip the coin once?

Binomial Experiment - Example 2

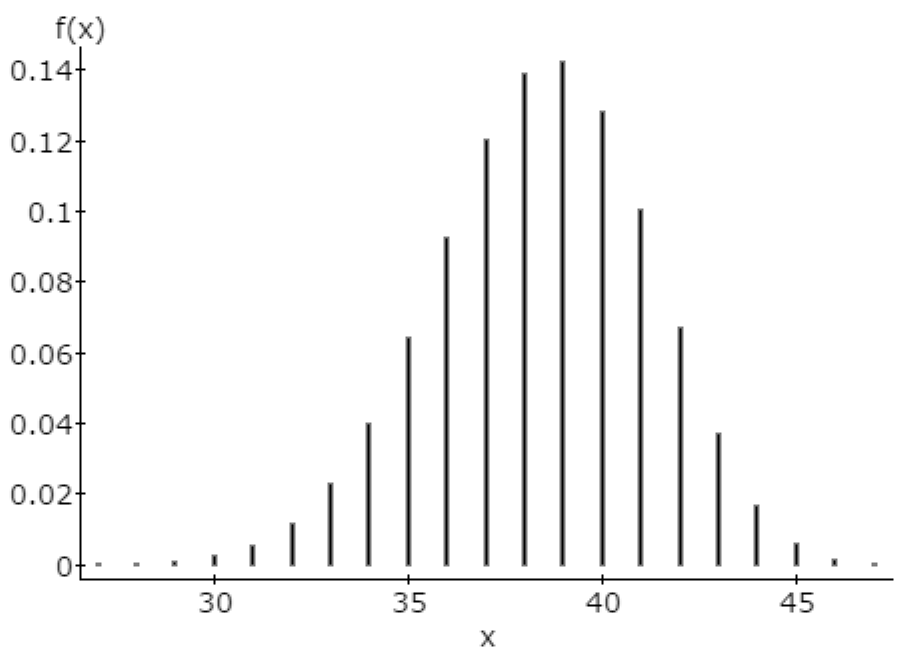
- After looking at some survey data you find that the probability that someone rates your attractiveness a two or higher is .80. Consider a class of 48 students.
- **n = 48**, p = 0.8, q = 1 - p = 1 – 0.8 = 0.2
- **Trials are independent** – one student's decision does not affect the others
- Let's go ahead and **assume identical trials** even though it can be argued that some people prefer different things

Binomial Experiment – Example 2

- Consider a class of 48 students.
  - Let X be the number of heads that occur
  - **n = 48**, p = 0.8, q = 1 - p = 1 – 0.8 = 0.2
  - $np = 48 * .8 = 38.4 \geq 15$ and $n(1-p) = 48 * (1 - .8) = 9.6 < 15$
  - Because $n(1-p) < 15$ we cannot say that the binomial is bell-shaped

Binomial Experiment – Example 2

- Consider a class of 48 students.
  - Because $n(1-p) < 15$ we cannot say that the binomial is bell-shaped



Binomial Experiment - Example 2

- Consider a class of 48 students.
- n = 48, p = 0.8, q = 1 - p = 1 – 0.8 = 0.2
- The probability that **exactly half** of the 48 students think you were at least a two out of ten

$$\mathrm{P(X = 24)} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \frac{48!}{24!\,(48-24)!} (.8)^{24} (.2)^{48-24} = \frac{48!}{24!\,24!} (.8)^{24}\,(.2)^{24}$$

$$= .00000255 = dbinom(24{,}48, .8)$$

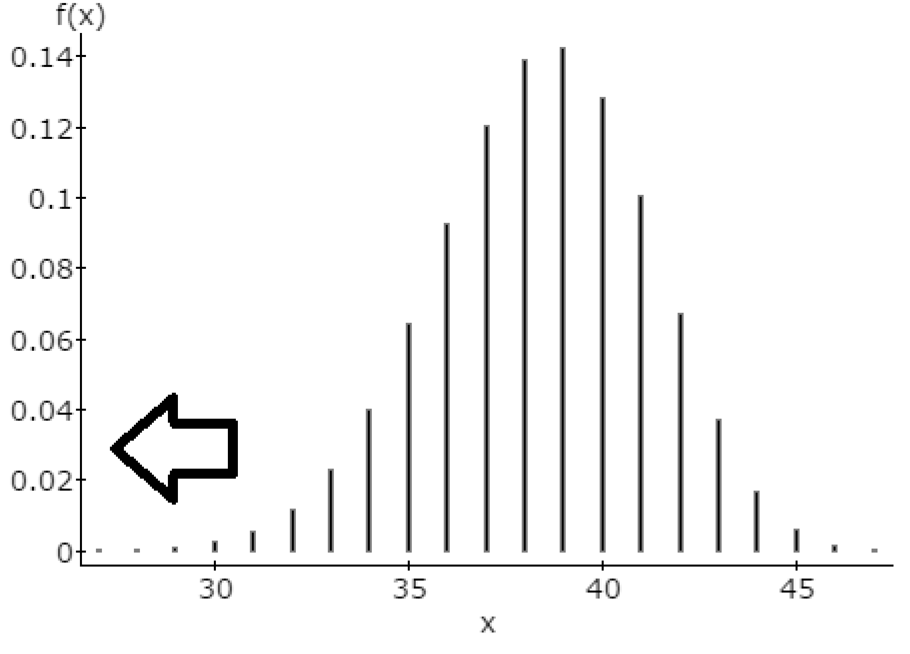
- This is an **almost impossible** event – we expect half of the class to think you were at least a two out of ten almost **0% of the time**

Binomial Experiment - Example 2

- Consider a class of 48 students.

$$P(X = 24) = .00000255$$

(Not visible because the probability is so small)



Binomial Experiment - Example 2

- Consider a class of 48 students.
- n = 48, p = 0.8, q = 1 - p = 1 – 0.8 = 0.2
- The probability that at least one of the students in your class think you were at least a two out of ten

$$\mathrm{P}(\mathrm{X} \geq 1) = P(X = 1) + P(X = 2) + \ldots P(X = 48)$$
$$= 1 - P(X = 0) = 1 - dbinom(0,48,.8)$$
$$= .999999999 \ldots$$

- This is an **almost certain** event – we expect at least half of the class to think you were at least a two out of ten **more than 99% of the time**

Helpful Rules for Discrete Distributions

- $\mathrm{P}(X < x) = P(X \leq x - 1)$
- $\mathrm{P}(X \geq x) = 1 - P(X < x)$
- $\mathrm{P}(X > x) = 1 - P(X \leq x)$
- $\mathrm{P}(x_1 < X < x_2) = P(X < x_2) - P(X \leq x_1)$
- $\mathrm{P}(x_1 < X \leq x_2) = P(X \leq x_2) - P(X \leq x_1)$
- $\mathrm{P}(x_1 \leq X < x_2) = P(X < x_2) - P(X < x_1)$
- $\mathrm{P}(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X < x_1)$

Mean and Variance For A Binomial

- So far we have found probabilities for the binomial distribution. This gave us the ability to check the feasibility of certain outcomes or groups of outcomes.
- Here, we find what to expect!

- **Expected Value = E(X) = Mean = $\boldsymbol{\mu}_x$** $= n * p$

- **Standard Deviation** $= \sigma_{\boldsymbol{x}} = \sqrt{n * p * q}$

Binomial Experiment – Example 1

- A fair one-cent piece is flipped three times
  - Let X be the number of heads that occur
  - n = 3
  - p = .50
  - q = 1 - p = 1 - .50 =.50
- $\boldsymbol{Mean} = n * p = 3 * .50 = 1.50$
- **On average, we expect** between 1 and 2 heads in three flips

Binomial Experiment – Example 1

- A fair one-cent piece is flipped three times
  - Let X bet the number of heads that occur
  - n = 3
  - p = .50
  - q = 1 - p = 1 - .50 =.50
- $\boldsymbol{Standard\ Deviation} = \sqrt{3 * .50 * .50} = .75$

Binomial Experiment – Example 1

- A fair one-cent piece is flipped three times
  - Let X bet the number of heads that occur
  - n = 3, p = .50, q = .50
- $\boldsymbol{Mean} = n * p = 3 * .50 = 1.50$
- $\boldsymbol{Standard\ Deviation} = \sqrt{3 * .50 * .50} = .75$

Binomial Experiment - Example 2

- Considering a class of 48 students.
- n = 48
- p = 0.8
- q = 1 - p = 1 – 0.8 = 0.2
- $\boldsymbol{Mean} = n * p = 48 * 0.80 = 38$
- So, on average we expect about 38 of the 48 students to think you’re at least a two out of ten.

Binomial Experiment - Example 2

- Considering a class of 48 students.
- n = 48
- p = 0.8
- q = 1 - p = 1 – 0.8 = 0.2
- $\boldsymbol{Standard\ Deviation} = \sqrt{48 * .80 * .20}$
  $= 2.7713$

Binomial Experiment - Example 2

- Considering a class of 48 students.
- n = 48, p = 0.8, q = 1 - p = 1 – 0.8 = 0.2
- $\boldsymbol{Mean} \; = \; n * p = 48 * 0.80 = 38$
- $\boldsymbol{Standard\ Deviation} = \sqrt{48 * .80 * .20}$
  $= 2.7713$
- Since we cannot say this binomial is bell-shaped we cannot use the empirical rule but we can use Chebyshev's Rule

Why don't I get this?

- Probabilities and expected values are much different than what we did in Chapter 2 where you found the sample mean by adding up values and dividing.
- Expected value in the sense of the binomial distribution is similar to the discrete distribution – it is what we would expect to see on average if we completed the binomial experiment infinitely many times
  - i.e. if I flipped a coin three times and kept track of how many heads I saw in each experiment over infinitely many experiments I would expect the average over all of those experiments to be $n*p=1.5$

Watch These!

- Binomial walk-through
  – https://www.youtube.com/watch?v=qIzC1-9PwQo

- TI-83/TI-84 BinomPDF
  – https://www.youtube.com/watch?v=6d1cKlEfqbQ

- TI-83/TI-84 BinomCDF
  – https://www.youtube.com/watch?v=uCZWamr75XE

Binompdf on your TI Calculator

- **Binomial P(X=x*)**
- <u>**INPUT:**</u>
- Press 2^{nd}
- Press VARS
- Scroll down using ↓ to highlight ‘A:binompdf(‘
- Press ENTER
- Type in your value for n
- Press ,
- Type in your value for p
- Press ,
- Type in your value for x*
- Press)
- Press ENTER
- <u>**OUTPUT:**</u> P(X=X*) = the numerical output.

Binomcdf on your TI Calculator

- Binomial P(X≤x*)
- **INPUT:**
- Press 2[nd]
- Press VARS
- Scroll down using ↓ to highlight 'B:binomcdf('
- Press ENTER
- Type in your value for n
- Press ,
- Type in your value for p
- Press ,
- Type in your value for x*
- Press)
- Press ENTER
- **OUTPUT:** P(X≤X*) = the numerical output.

Binomial on StatCrunch

- Open StatCrunch
- Stat→ Calculators→Binomial → Enter n and p→ insert whichever probability statement you need→Compute

The Binomial Distribution

- We look at a categorical variable with two outcomes
  - We consider one a success and zero a failure

x		P(x)
Success (denoted as 1)	This is what we're interested in, even if it isn't particularly successful in the sense of the English word	p = Probability of a 'success'
Failure (denoted as 0)	This is the other case – what we aren't interested in ,even if it isn't particularly a failure in the sense of the English word	q = Probability of a 'failure' = 1- p

Binomial Distribution

Assumptions	1. It consists of **n trials** with **binary output** They are denoted 1 or 0, or success and failure 2. The probability of success on each trial is the same The trials are **identical** 3. The outcome of one trial does not affect the outcome of another trial The trials are **independent**
Formula	$P(X = x) = \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$ We will say that the binomial is bell-shaped if $n * p \geq 15\ AND\ n * (1 - p) \geq 15$
Expected value of Binomial X	$\mu_x = E(X) = n * p$
Standard Deviation of Binomial X	$\sigma_{\boldsymbol{x}} = \sqrt{n * p * q}$